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Reg. No. : ....

Name : ....

Fifth Semester B.Sc. Degree Examination, December 2022

First Degree Programme under CBCSS

Mathematics

Core Course

MM 1541 : REAL ANALYSIS I

(2018 Admission Onwards)

Time : 3 Hours Max. Marks : 80

SECTION I

Answer **all** the questions.

1. Define absolute value function $|x|$.
2. Define *least upper bound* for a set $A \subseteq R$.
3. Find $\inf(A)$ if $A = \left\{\frac{1}{n}, n \in \mathbb{N}\right\}$
4. Find $\lim_{n\to\infty}\left(\frac{3n+2}{2n-2}\right)$.
5. Define the limit of a real sequence.
6. Write the harmonic series. Is it convergent?
7. Define Cauchy sequence.
8. Let $\in = \frac{1}{2}$ and a = 3, then find the $\in$ - neighborhood, $V_{\in}(a)$ of a.
9. Define closed set in $\mathbb{R}$. Write an example for closed set.
10. Define perfect set in $\mathbb{R}$. **(10 × 1 = 10 Marks)**

P.T.O.

SECTION II

Answer any **eight** questions

11. Show that, two real numbers a and b are equal if and only if for every real number $\in > 0$ it follows that $|a-b| < \in$.

12. Prove that $\sqrt{3}$ is irrational.

13. Using triangle inequality prove the inequality $\left||a|-|b|\right| \leq |a-b|$.

14. Let $A \subseteq R$ be bounded above, and let $c \in R$. Define the set $c+A$ by $c+A = \{c+a : a \in A\}$. Show that $\sup(c+A) = c + \sup A$.

15. Show that $\lim\left(\frac{n+1}{n}\right) = 1$.

16. Show that every convergent sequence is bounded.

17. Let $\lim a_n = a$, and $\lim b_n = b$, then prove that $\lim(a_n + b_n) = a + b$.

18. Show that if $(b_n) \to b$, then the sequence of absolute values $|b_n|$ converges to $|b|$.

19. Write the examples for

    (a) Sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;

    (b) A convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges.

20. Prove that the sequence defined by $x_1 = 3$ and $x_{n+1} = \frac{1}{4 - x_n}$ converges.

21. Show that, sub sequences of a convergent sequence converge to the same limit as the original sequence.

22. Show that, every convergent sequence is a Cauchy sequence.

23. Let $a \in A$. Prove that a is an isolated point of A if and only if there exists an $\in -$ neighborhood $V_\in(a)$ such that $V_\in(a) \cap A = \{a\}$.

24. Show that if K is compact, then $\sup K$ and $\inf K$ both exist and are elements of K.

25. Prove that cantor set is a Compact set.

26. Let E is nowhere-dense in $\mathbb{R}$, then show that the complement of $\overline{E}$ is dense in $\mathbb{R}$.

(8 × 2 = 16 Marks)

SECTION III

Answer any **six** questions

27. Assume $s \in R$ is an upper bound for a set $A \subseteq R$. Then, show that $s = \sup A$ if and only if, for every choice of $\in > 0$, there exists an element $a \in A$ satisfying $s - \in < 0$.

28. State and prove Nested Interval Property.

29. Show that the set of rational numbers, $\mathbb{Q}$ is countable.

30. Given any set A, prove that there does not exist a function $f : A \to P(A)$ that is onto.

31. Assume $\lim a_n = a$ and $\lim b_n = b$, = b, then show that

    (a) If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.

    (b) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.

32. Show that; if $x_n \leq y_n \leq z_n \; \forall n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$.

33. If a sequence is monotone and bounded, then show that it is convergent.

34. Show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

35. Prove that, a point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.

36. For any $A \subseteq \mathbb{R}$, Show that the closure $\overline{A}$ is a closed set and is the smallest closed set containing A.

37. If $K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \ldots$ is a nested sequence of nonempty compact sets, then show that, the intersection $\bigcap_{n=1}^{\infty} K_n$ is not empty.

38. Prove that the set of real numbers $\mathbb{R}$ cannot be written as the countable union of nowhere-dense sets.

(6 × 4 = 24 Marks)

SECTION IV

Answer any **two** questions

39. Prove that

(a) Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.

(b) Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $\frac{1}{n} < y$.

(c) For every two real numbers a and b with $a < b, \exists$ a rational number r, satisfying $a < r < b$.

40. (a) Show that, there exists a real number $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.

(b) Find *suprema* and *infima* of the following sets.

(i) $\{n \in \mathbb{N} : n^2 < 10\}$

(ii) $\{\frac{n}{m} : m, n \in \mathbb{N}$ with $m + n \leq 10\}$

41. (a) State and prove Bolzano-Weierstrass theorem.

(b) Prove or disprove: Every Bounded sequence of real number are convergent.

42. Show that

(a) Cauchy sequences are bounded.

(b) A sequence converges if and only if it is a Cauchy sequence.

43. (a) State and prove Cauchy Criterion for Series.

(b) Using Alternating Series Test, test the convergence of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{7+2n}$.

44. (a) Show that

(i) The union of an arbitrary collection of open sets is open.

(ii) The intersection of a finite collection of open sets is open.

(b) Let $B = \left\{ \frac{(-1)^n n}{n+1}, n \in \mathbb{N} \right\}$

(i) Find the limit points of B.

(ii) Does B contain any isolated points?

(iii) Find closure of B, $\overline{B}$.

(2 × 15 = 30 Marks)