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Reg. No. : ....

Name : ....

Third Semester B.Sc. Degree Examination, January 2023

First Degree Programme under CBCSS

Mathematics

Core Course

MM 1341 : ELEMENTARY NUMBER THEORY AND CALCULUS – I

(2019 Admission Onwards)

Time : 3 Hours Max. Marks : 80

SECTION – A

Answer **all** questions.

1. For every positive integer n, find n consecutive integers that are composite numbers.

2. Prove that there are infinitely many primes.

3. State Dirichlet's theorem.

4. State the Pigeonhole principle.

5. If $r(t)= t^2 i + e^t j - (2 - \cos \pi t)k$, find $r'(t)$.

6. Prove that a straight line has zero curvature at every point.

7. Evaluate: $\int_0^2 r(t)\,dt$, where $r(t) = 2ti + 3t^2 j$.

8. If f is a function of x, y and z, what is the gradient of f?

9. State the chain rules for partial derivatives.

10. Let $f(x, y) = y^2 e^x + y$. Evaluate f_{xyy}.

(10 × 1 = 10 Marks)

SECTION – B

Answer any **eight** questions.

11. Show that every composite number n has a prime factor $\leq \lfloor \sqrt{n} \rfloor$.

12. Using recursion, evaluate, (18, 30, 60, 73,132).

13. Derive a necessary and sufficient condition for two positive integers to be relatively prime.

14. Prove that $(a, b) = (a, a - b)$.

15. Find the number of positive integers ≤ 2076 and divisible by neither four nor five.

16. Write a short note on twin primes.

17. If $r(t)$ is a differentiable vector – valued function in 2-space or 3-space and $\|r(t)\|$ is constant for all t, then show that $r(t)$ and $r'(t)$ are orthogonal vectors for all t.

18. State any two rules of integration of vector valued functions.

19. State the Newton's laws of universal gravitation.

20. Show that the circle of radius a which centred at the origin has constant curvature $\frac{1}{a}$.

21. Evaluate the unit tangent vector to the graph of $r(t) = t^2 i + t^3 j$ at the point where $t = 2$.

22. Estimate an equation for the tangent plane and parametric equations for the normal line to the surface $z = x^2 y$ at the point (2, 1, 4).

23. Find the directional derivative of $f(x, y, z) = x^2 y - yz^3 + z$ at $(1, -2, 0)$ in the direction of the vector $a = 2i + j - 2k$.

24. Evaluate $f_x(1,3)$ and $f_y(1,3)$ by finding $f_x(x, y)$ and $f_y(x, y)$ where $f(x, y) = 2x^3 y^2 + 2y + 4x$.

25. Write the steps to find the absolute extrema of a continuous function f of two variables on a closed and bounded set R.

26. Prove that $f(x, y) = x^2 + y^2$ is differentiable at the origin.

(8 × 2 = 16 Marks)

SECTION – C

Answer any **six** questions.

27. Prove that there are $3\lfloor n/2 \rfloor$ primes in the range n through $n!$, where $n \geq 4$.

28. Let e denote the highest power of 2 that divides $n!$ and b the number of 1s in the binary representation of n. Then show that $n = e + b$.

29. Show that the gcd of the positive integers a and b is a linear combination of a and b.

30. Show that 3, 5 and 7 are the only three consecutive odd integers that are primes.

31. Find parametric equations of the tangent line to the circular helix $x = \cos t$, $y = \sin t, z = t$ where $t = t_0$, and use that result to find parametric equations for the tangent line at the point $t = \pi$.

32. Suppose that a particle moves along a circular helix in 3-space so that its position vector at time t is $r(t) = (4\cos \pi t)i + (4\sin \pi t)j + t\,k$. Find the distance travelled and the displacement of the particle during time interval $1 \leq t \leq 5$.

33. Find the curvature of the ellipse with vector equation $r = 2\cos t\, i + 2\sin t\, j$, $(0 \leq t \leq 2\pi)$ at the end points of the major and minor axes.

34. Derive Kepler's third law.

35. Find the slope of the sphere $x^2 + y^2 + z^2 = 1$ in the y- direction at the points $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$.

36. Verify whether the function $z = e^x \sin y + e^y \cos x$ satisfies the equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

37. Locate all relative extrema and saddle points of $f(x, y) = 3x^2 - 2xy + y^2 - 8y$.

38. Use appropriate forms of the chain rule to find $\frac{\partial w}{\partial \rho}$ and $\frac{\partial w}{\partial \theta}$ where $w = x^2 + y^2 - z^2, x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta$ and $z = \rho \cos\phi$.

(6 × 4 = 24 Marks)

SECTION – D

Answer any **two** questions.

39. (a) Prove that there is no polynomial $f(n)$ with integral coefficients that will produce primes for all integers n.

    (b) Find the general solution of the LDE $6x + 8y + 12z = 10$.

40. (a) State and prove the division algorithm.

    (b) Show that the number of leap years l after 1600 and not exceeding a given year y is given by $l = \lfloor y/4 \rfloor - \lfloor y/100 \rfloor + \lfloor y/400 \rfloor - 388$.

41. A shell fired from a cannon has a muzzle speed of 800 ft/s. The barrel makes an angle of 45° with the horizontal and, for simplicity, the barrel opening is assumed to be at ground level.

    (a) Find parametric equations for the shell's trajectory.

    (b) How high does the shell rise?

    (c) How far does the shell travel horizontally?

    (d) What is the speed of the shell at its point of impact with the ground?

42. Suppose that a particle moves through 3-space so that its position vector at time t is $r(t) = ti + t^2 j + t^3 k$.

    (a) Find the scalar tangential and normal components of acceleration at time t.

    (b) Find the scalar tangential and normal components of acceleration at time $t = 1$.

    (c) Find the vector tangential and normal components of acceleration at time $t = 1$

    (d) Find the curvature of the path at the point where the particle is located at time $t = 1$

43. (a) A heat - seeking particle is located at the point (2, 3) on a flat metal plate whose temperature at a point (x, y) is $T(x, y) = 10 - 8x^2 - 2y^2$. Find an equation for the trajectory of the particle if it moves continuously in the direction of maximum temperature increase.

(b) The length, width and height of a rectangular box are measured with an error of at most 5%. Use a total differential to estimate the maximum percentage error that results in these quantities are used to calculate the diagonal of the box.

44. Find the points on the sphere $x^2 + y^2 + z^2 = 36$ that are closest to and farthest from the point $(1, 2, 2)$.

(2 × 15 = 30 Marks)
