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U – 2378

Reg. No. : ....

Name : ....

Fifth Semester B.Sc. Degree Examination, December 2024

First Degree Programme under CBCSS

Mathematics

Core Course

MM 1543 : ABSTRACT ALGEBRA – GROUP THEORY

(2018 Admission onwards)

Time : 3 Hours

Max. Marks : 80

SECTION – I

All the **first ten** questions are **compulsory**. They carry **1** mark each.

1. Check whether the set {0,1, 2, 3} is a group under multiplication modulo 4.

2. In a group G. show that there is only one identity element.

3. Find the subgroup of Z_{30} of order 10.

4. Express (1 2 3 4 5) as product of 2-cycles.

5. Check whether the mapping $\varphi:(R,+)\to(R,+)$ defined by $\varphi(x)=x^3$ is an isomorphism.

6. Find an automorphism of the group of complex numbers under addition.

7. Let $H=\{0,\pm 3,\pm 6,\pm 9,\ldots\ldots\ldots\}$. Find all left cosets of H in Z.

P.T.O.

8. Show that all groups of order 25 is Abelian.

9. Let $\varphi : R^* \to R^*$ be defined by $\varphi(x) = |x|$. Find $Ker\varphi$.

10. How many Abelian groups (upto isomorphism) are there of order 15.

(10 × 1 = 10 Marks)

SECTION – II

Answer **any eight** questions. These questions carry **2** marks each .

11. Let G be an Abelian group. Show that $H = \{x \in G : |x| \text{ is finite}\}$ is a subgroup of G.

12. For group elements a, b, show that $(ab)^{-1} = b^{-1}a^{-1}$.

13. Find all generators of the subgroup of order 9 in Z_{36}.

14. Show that S_3 is a non-Abelian group.

15. What is the order of the permutation $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 4 & 6 & 3 \end{bmatrix}$.

16. Show that there is no isomorphism from Q, the group of rational numbers under addition to Q^* the group of non zero rational numbers under multiplication.

17. Show that every group of prime order is cyclic.

18. Show that a group of order 75 can have atmost one subgroup of order 25.

19. Show that the group $SL(2, R)$ is a normal subgroup of $GL(2, R)$.

20. Let φ be a homomorphism from a group G to a group G'. Show that $\varphi(a) = \varphi(b)$ if and only if $a\,Ker\varphi = b\,Ker\varphi$.

21. Show that $Z/\langle n \rangle \approx Z_n$.

22. Show that center of a group G is a subgroup of G.

(8 × 2 = 16 Marks)

SECTION – III

Answer **any six** questions. These questions carry **4** marks each.

23. Let a be an element of order n in a group and let k be positive integer. Show that $\langle a^k \rangle = \langle a^{\gcd(n,k)} \rangle$ and $|a^k| = n / \gcd(n,k)$.

24. Explain the dihedral group D_n of order $2n$.

25. Determine the number of elements in S_7 of order 12.

26. Compute $Aut(Z_{10})$.

27. Let φ be a homomorphism from a group G onto a group G'. Prove that $G = \langle a \rangle$ if and only if $G' = \langle \varphi(a) \rangle$.

28. For any two finite subgroup H and K, show that $|HK| = |H||K| / |H \cap K|$.

29. Let G be a group and let H be a normal subgroup of G. Show that the set $G/H = \{aH : a \in G\}$ is a group under the operation $(aH)(bH) = abH$.

30. Let G be a group and let $Z(G)$ be the center of G. If $G/Z(G)$ is cyclic, show that G is Abelian.

31. Show that a group of order 35 is cyclic.

(6 × 4 = 24 Marks)

SECTION – IV

Answer **any two** questions. These questions carry **15** marks each.

32. (a) Prove that if a is the only element of order 2 in a group, then a lies in the center of the group.

    (b) Let G be a group and let $a \in G$. If a has infinite order. then show that $a^i = a^j$ if and only if $i = j$. If a has finite order n, then show that $\langle a \rangle = \{e, a, a^2, \ldots a^{n-1}\}$ and $a^i = a^j$ if and only if n divides $i - j$.

33. (a) If $\varepsilon = \beta_1\beta_2......\beta_r$, where β's are 2-cycles, then show that r is even.

(b) Show that the group of rotations of a cube is isomorphic to S_4.

34. If a group G is the internal direct product of a finite number of subgroups $H_1, H_2, \ldots \ldots H_n$, then show that G is isomorphic to the external direct product of $H_1, H_2, \ldots \ldots H_n$.

35. (a). Let G be a finite Abelian group of order $p^n m$, where p is a prime that does not divide m. Show that $G \approx H \times K$, where $H = \{X \in G : x^{p^n} = e\}$ and $K = \{x \in G : x^m = e\}$.

(b) State and prove first isomorphism theorem.

(2 × 15 = 30 Marks)